International Journal of Theoretical Physics, Vol. 36, No. 5, 1997

Uniqueness of the Bethe Ansatz of the *XXZ* Chain and Statistical Interaction

Dong Mi,[1] Fuzhong Yang,[1] and Buoyuan Hou[1]

Received September 13, 1996

We prove that there is only one form of the Bethe ansatz with the general *XXZ* chain. We give an explanation of the *XXZ* chain in terms of statistical interactions.

1. INTRODUCTION

Alcaraz *et al.* (1987) derived the Bethe ansatz equations for the free *XXZ* chain with an arbitrary surface field at each end of the chain. The free *XXZ* chain is a very important model, for it has the symmetry of the quantum algebra $su_q(2)$ (Pasquier and Saleur, 1990). They start from the general quantum *XXZ* model with the following Hamiltonian:

$$H_{XXZ} = -\frac{1}{2}\left[\sum_{j=1}^{L-1} (\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z) + p\sigma_1^z + p'\sigma_L^z\right] \quad (1.1)$$

where Δ, p, and p' are arbitrary constants. Considering that this Hamiltonian commutes with the total spin operator $\Sigma\,\sigma^z$, the number of down spins n is a good quantum number. They investigate the following eigenvalue equation:

$$H|n\rangle = E|n\rangle \quad (1.2)$$

where

$$|n\rangle = \sum (x_1, \ldots, x_n)|x_1, \ldots, x_n\rangle \quad (1.3)$$

Here the $x_1, \ldots, x_n$ denote the locations of the down spins on the chain, and

[1] Department of Physics, Graduate School, Academia Sinica, Beijing 100039, China.

the summation extends over all sets of the n increasing integers varying between 1 and L:

$$1 \leq x_1 < x_2 < \cdots < x_n \leq L \tag{1.4}$$

For general n, they assume that the ansatz for the wavefunction is

$$f(x_1, \ldots, x_n) = \sum_P \epsilon_P A(k_1, \ldots, k_n) e^{i(k_1 x_1 + \ldots, + k_n x_n)} \tag{1.5}$$

where the summation extends over all permutations and negations of $k_1, \ldots, k_n$ and ϵ_P changes sign at each such 'mutation.' From this assumption, they obtain the Bethe ansatz equations which the momenta $k_1, \ldots, k_n$ satisfy.

From a more general point of view, we will take for ϵ_P the values $e^{i\varphi_1}$ and $e^{i\varphi_2}$ when negations and permutations appear one time, respectively. Here φ_1 and φ_2 are arbitrary values between 0 and 2π. From this more general assumption we will obtain the same Bethe ansatz equations as Alcaraz *et al.* and then prove the uniqueness of the Bethe ansatz corresponding the *XXZ* quantum chain.

Although we could not apply the point of view of anyons directly to the Bethe ansatz, we can give an explanation of statistical interactions. Haldane (1991) introduced a notion of fractional statistics which is independent of the dimension of space. According to Haldane, consider a system with a total number of particles $N = \Sigma N_j$, with N_j the number of particles of the species j, which is confined to a finite region of matter. Now if we add a particle of the species i into the system without changing the size and boundary conditions of the system, and let the coordinates of the N particles of the original system be fixed, then the wave function of the new $(N + 1)$-body system can be expanded in a basis of wave functions of the added particle. The crucial point is that the number D_i of available single-particle states in this basis for a particle of species i generally is not a constant; it may depend on the particle number N_j of all species in the original system. Haldane defined statistical interactions through the linear relation

$$\Delta D_i = -\sum_j g_{ij} \Delta N_j \tag{1.6}$$

where $\{\Delta N_j\}$ is a set of allowed changes of the particle number. Equation (1.6) can be rewritten as

$$D_i(\{N_j\}) + \sum_j g_{ij} N_j = G_i^0 \tag{1.7}$$

with $G_i^0 = D_i(\{0\})$ a constant, which is interpreted as the number of available single-particle states when no particle is in the system. Namely, G_i^0 are the bare numbers of single-particle states. From this point of view, Bernard and

Wu (1994) explained the Bethe ansatz equations of the N-body integrable model with periodic boundary condition. In the following discussion, we will explain how the Bethe ansatz equations of the general quantum XXZ chain can be reinterpreted in such way that they code a statistical interaction among particles.

2. BETHE ANSATZ

For general n we assume that the Bethe ansatz for the wavefunction is

$$f(x_1, \ldots, x_n) = \sum_P \epsilon'_P A(k_1, \ldots, k_n) e^{i(k_1 x_1 + \cdots + k_n x_n)} \tag{2.1}$$

where the summation extends over all permutations and negations of $k_1, \ldots, k_n$ and ϵ'_P takes the values $e^{im\varphi_1}$ and $e^{im\varphi_2}$ when negations and permutations appear m times, respectively.

2.1. $n = 1$

For one down spin on the chain, the eigenvalue equation (1.2) gives

$$\begin{aligned} Ef(x) = {} & -f(x-1) - f(x+1) \\ & - \tfrac{1}{2}[(L-5)\Delta + p + p']f(x), \qquad x = 2, \ldots, L-1 \end{aligned} \tag{2.2}$$

At the boundaries we get slightly different equations:

$$Ef(1) = -f(2) - \tfrac{1}{2}[(L-3)\Delta - p + p']f(1) \tag{2.3a}$$

$$Ef(L) = -f(L-1) - \tfrac{1}{2}[(L-3)\Delta + p - p']f(L) \tag{2.3b}$$

According to equation (2.1), we can substitute the solution

$$f(x) = A(k)e^{ikx} + e^{i\varphi_1}A(-k)e^{-ikx} \tag{2.4}$$

in equation (2.2) and obtain the eigenvalue

$$E = -2\cos k - \tfrac{1}{2}[(L-5)\Delta + p + p'] \tag{2.5}$$

If we want equation (2.2) to be valid for $x = 1$ and $x = L$ also, where $f(0)$ and $f(L+1)$ are defined by (2.4), then from (2.2) and (2.3) we get the end conditions

$$f(0) = (\Delta - p)f(1) \tag{2.6a}$$

$$f(L+1) = (\Delta - p')f(L) \tag{2.6b}$$

Defining the functions $\alpha(k)$ and $\beta(k)$ by

$$\alpha(k) = 1 + (p - \Delta)e^{-ik} \tag{2.7a}$$

$$\beta(k) = [1 + (p' - \Delta)e^{-ik}]e^{i(L+1)k} \tag{2.7b}$$

and substituting (2.4) in (2.6), we obtain

$$A(k)\alpha(-k) + e^{i\varphi_1}A(-k)\alpha(k) = 0 \tag{2.8a}$$

$$A(k)\beta(k) + e^{i\varphi_1}A(-k)\beta(-k) = 0 \tag{2.8b}$$

Compatibility between (2.8a) and (2.8b) yields

$$\alpha(k)\beta(k) = \alpha(-k)\beta(-k) \tag{2.9}$$

Given the compatibility relation (2.9), the solution of (2.8) for $A(k)$ is

$$A(k) = \beta(-k) \tag{2.10a}$$

$$A(-k) = -e^{i\varphi_1} = e^{i(\pi-\varphi_1)}\beta(k) \tag{2.10b}$$

where $A(k)$ is not invariant under $k \leftrightarrow -k$.

2.2. $n = 2$

For two down spins on the chain, we obtain the eigenvalue equation

$$Ef(x_1, x_2) = -f(x_1 - 1, x_2) - f(x_1 + 1, x_2) - f(x_1, x_2 - 1) - f(x_1, x_2 + 1) - \tfrac{1}{2}[(L - 9)\Delta + p + p']f(x_1, x_2) \tag{2.11}$$

We now also get the usual 'meeting condition' that arises because the two down spins may be neighbors,

$$f(x_1, x_1) + f(x_1 + 1, x_1 + 1) - 2\Delta f(x_1, x_1 + 1) = 0 \tag{2.12}$$

As in the case $n = 1$, we have two conditions to be satisfied at the free ends of the chain

$$f(0, x_2) = (\Delta - p)f(1, x_2) \tag{2.13a}$$

$$f(x_1, L + 1) = (\Delta - p')f(x_1, L) \tag{2.13b}$$

Guided by the $n = 1$ case, we consider the ansatz

$$\begin{aligned} f(x_1, x_2) &= \sum_P \epsilon'_P A(k_1, k_2)e^{i(k_1x_1+k_2x_2)} \\ &= A(k_1, k_2)e^{i(k_1x_1+k_2x_2)} + A(k_2, k_1)e^{i(k_2x_1+k_1x_2)}e^{i\varphi_2} \\ &\quad + A(-k_1, k_2)e^{i(-k_1x_1+k_2x_2)}e^{i\varphi_1} + A(k_1, -k_2)e^{i(k_1x_1-k_2x_2)}e^{i\varphi_1} \\ &\quad + A(-k_2, k_1)e^{i(-k_2x_1+k_1x_2)}e^{i(\varphi_1+\varphi_2)} + A(k_2, -k_1)e^{i(k_2x_1-k_1x_2)}e^{i(\varphi_1+\varphi_2)} \\ &\quad + A(-k_1, -k_2)e^{i(-k_1x_1-k_2x_2)}e^{i(2\varphi_1)} + A(-k_2, -k_1)e^{i(-k_2x_1-k_1x_2)}e^{i(2\varphi_1+\varphi_2)} \end{aligned} \tag{2.14}$$

Substituting this ansatz in (2.11), we obtain the eigenvalue

$$E = -2\cos k_1 - 2\cos k_2 - \tfrac{1}{2}[(L-9)\Delta + p + p'] \tag{2.15}$$

Defining the function $s(k_1, k_2)$

$$s(k_1, k_2) = 1 - 2\Delta e^{ik_2} + e^{i(k_1+k_2)} \tag{2.16}$$

we find the sufficient and necessary conditions for equations (2.12) and (2.13) to be satisfied:

$$A(k_1, k_2)s(k_1, k_2) + e^{i\varphi_2}A(k_2, k_1)s(k_2, k_1) = 0 \tag{2.17a}$$

$$A(k_1, k_2)\alpha(-k_1) + e^{i\varphi_1}A(-k_1, k_2)\alpha(k_1) = 0 \tag{2.17b}$$

$$A(k_1, k_2)\beta(k_2) + e^{i\varphi_1}A(k_1, -k_2)\beta(-k_2) = 0 \tag{2.17c}$$

together with nine other equations. The compatibility of these equations gives the following Bethe ansatz equations:

$$\frac{\alpha(k_1)\beta(k_1)}{\alpha(-k_1)\beta(-k_1)} = \frac{B(-k_1, k_2)}{B(k_1, k_2)} \tag{2.18a}$$

$$\frac{\alpha(k_2)\beta(k_2)}{\alpha(-k_2)\beta(-k_2)} = \frac{B(-k_2, k_1)}{B(k_2, k_1)} \tag{2.18b}$$

where $B(k, k')$ is given by

$$B(k, k') = s(k, k')s(k', -k) \tag{2.19}$$

Following Alcaraz *et al.* (1987), the coefficients in equation (2.14) now are given by

$$A(k_1, k_2) = \beta(-k_1)\beta(-k_2)B(-k_1, k_2)e^{-ik_2} \tag{2.20a}$$

$$A(-k_1, k_2) = \beta(k_1)\beta(-k_2)B(k_1, k_2)e^{-ik_2}e^{i(\pi-\varphi_1)} \tag{2.20b}$$

$$A(k_2, k_1) = \beta(-k_2)\beta(-k_1)B(-k_2, k_1)e^{-ik_1}e^{i(\pi-\varphi_2)} \tag{2.20c}$$

$$\vdots$$

2.3. General *n*

The above discussion can be generalized to arbitrary values of n. The coefficients in the wavefunction are given by

$$A(k_1, \ldots, k_n) = \sum_{j=1}^{n} \beta(-k_j) \sum_{1\le j<l\le n} B(-k_j, k_l)e^{-ik_l} \tag{2.21a}$$

$$A(\epsilon_{P_1}k_{P_1}, \ldots, \epsilon_{P_n}k_{P_n}) = \sum_{j=1}^{n} \beta(-k_j) \sum_{1\le j<l\le n} B(-k_j, k_l)e^{-ik_l}e^{i[(m+m')\pi - m\varphi_1 + m'\varphi_2]} \tag{2.21b}$$

where from order $(k_1, \ldots, k_n)$ to order $(\epsilon_{P_1} k_{P_1}, \ldots, \epsilon_{P_1} k_{P_n})$, there are m times negations and m' times permutations, respectively, and ϵ_{P_i} take values ± 1. Substituting these coefficients in equation (2.1), we find that the form of Bethe ansatz corresponding to the *XXZ* chain is the same as that of Alcaraz *et al.* (1987). The above proof indicates that there exists a unique form of Bethe ansatz for the general quantum *XXZ* chain.

The parameters $k_1, \ldots, k_n$ satisfy the following Bethe ansatz equations:

$$\frac{\alpha(k_j)\beta(k_j)}{\alpha(-k_j)\beta(-k_j)} = \sum_{l=1\neq j}^{n} \frac{B(-k_j, k_l)}{B(k_j, k_l)}, \qquad j = 1, \ldots, n \tag{2.22}$$

The eigenvalues E are given by

$$E = -\frac{1}{2}[(L-1)\Delta + p + p'] - 2\sum_{j=1}^{n} (\cos k_j - \Delta) \tag{2.23}$$

3. THE STATISTICAL INTERACTIONS AND THE BETHE ANSATZ

Taking the logarithm of equation (22), we obtain that the parameters k_j $(j = 1, \ldots, n)$ satisfy

$$2(L+1)k_j - \Phi(p) - \Phi(p') = 2\pi l_j + \sum_{l=1,l\neq j}^{n} [\Theta(k_j, -k_l) + \Theta(k_j, k_l)] \tag{3.1}$$

where

$$\Phi(q) = -2\tan^{-1}\left[\frac{(\Delta - q)\sin k_j}{1 - (\Delta - q)\cos k_j}\right] \tag{3.2}$$

and

$$\Theta(k, k') = -2\tan^{-1}\left[\frac{\Delta \sin(k - k')/2}{\cos(k + k')/2 - \Delta\cos(k - k')/2}\right] \tag{3.3}$$

and $\{l_j\}$ $(j = 1, \ldots, n)$ is a set of integers or half integers, depending on n being odd or even, which can be chosen as the quantum numbers labeling the eigenstates instead of the momenta $\{k_j\}$. Consider the cases when these l_j are all different. The ground state corresponds to an equidistribution of the integral quantum numbers $\{l_j\}$ in an interval centered around the origin: $l_{j+1} - l_j = 1$. The excited states correspond to the nonequidistributions of $\{l_j\}$: $l_{j+1} - l_j = 1 + M_j^{\text{h}}$, where M_j^{h} is the number of holes, i.e., unoccupied integer numbers between l_{j+1} and l_j. The description of the states by the quantum numbers $\{l_j\}$ can be called a fermionic description, for there cannot be two integers taking the same value. In the following, we turn to the momentum description. Equation (3.1) can be rewritten as

$$2\pi l_j = 2(L+1)k - \sum_{k\neq k', k\neq -k'} [\Theta(k, -k') + \Theta(k, k') + \Phi(p) + \Phi(p')] \quad (3.4)$$

Following Yang and Yang (1969), let

$$Lh(k) = (L+1)k - \sum_{k\neq k', k\neq -k'} [\Theta(k, -k') + \Theta(k, k') + \Phi(p) + \Phi(p'] \quad (3.5)$$

The values of k where $Lh(k) = \pi l_j$ are particles. The values of k where $Lh(k) = \pi J$ ($J = n/l_j$) can be defined as holes.

In the thermodynamic limit, we introduce the density distribution of holes as well as that of particles:

$$L\rho(k)\, dk = \text{number of particles in } dk$$

$$L\rho_h(k)\, dk = \text{number of holes in } dk \quad (3.6)$$

Since $dh(k) = \pi/L$ (number of particles and holes in dk) $= \pi[\rho(k) + \rho_h(k)]\, dk$, then

$$\frac{dh(k)}{dk} = \pi[\rho(k) + \rho_h(k)] \equiv \pi\rho_t(k) \quad (3.7)$$

In the thermodynamic limit, equation (3.5) becomes

$$h(k) = k - \frac{1}{2}\int_{\infty}^{\infty} [\Theta(k, -k') + \Theta(k, k') + \Phi(p) + \Phi(p']\rho(k')\, dk' \quad (3.8)$$

Differentiation with respect to k gives

$$\begin{aligned}
\pi\rho_t(k) &= \pi[\rho(k) + \rho_h(k)] \\
&= 1 + \int_{-\infty}^{\infty} \Bigg\{\Delta(\cos k' - \Delta) \\
&\quad \times \left[\frac{1}{1 + \cos(k - k') - 2\Delta(\cos k + \cos k') + 2\Delta^2}\right. \\
&\quad \left. + \frac{1}{1 + \cos(k + k') - 2\Delta(\cos k + \cos k') + 2\Delta^2}\right] \\
&\quad + \frac{(\Delta - p)[\cos k_j - (\Delta - p)]}{1 - 2(\Delta - p)\cos k_j + (\Delta - p)^2} \\
&\quad + \frac{(\Delta - p')[\cos k_j - (\Delta - p')]}{1 - 2(\Delta - p')\cos k_j + (\Delta - p')^2}\Bigg\}\rho(k')\, dk' \quad (3.9)
\end{aligned}$$

Comparison with the fractional statistics equation (1.7) shows that they are identical provided we make the following identification [let $\rho^0(k) = 1/\pi$]

$$\begin{aligned} G_i^0/L &\leftrightarrow \rho^0(k) \\ N_i/L &\leftrightarrow \rho(k) \\ D_i(\{N_j\}) &\leftrightarrow \rho_h(k) \end{aligned} \tag{3.10}$$

with the discrete sum replaced by the integral over momentum. The last equation shows clearly that the hole density $\rho_h(k)$ represents the density of available states for an additional particle to be added. Furthermore, we can obtain the formula for the statistical interaction in the momentum space. If we define

$$\begin{aligned} g(k, k') = \delta(k - k') - \frac{1}{\pi}\Bigg\{ & \Delta(\cos k' - \Delta) \\ & \times \left[\frac{1}{1 + \cos(k - k') - 2\Delta(\cos k + \cos k') + 2\Delta^2} \right. \\ & \left. + \frac{1}{1 + \cos(k + k') - 2\Delta(\cos k + \cos k') + 2\Delta^2}\right] \\ & + \frac{(\Delta - p)[\cos k_j - (\Delta - p)]}{1 - 2(\Delta - p)\cos k_j + (\Delta - p)^2} \\ & + \frac{(\Delta - p')[\cos k_j - (\Delta - p')]}{1 - 2(\Delta - p')\cos k_j + (\Delta - p')^2}\Bigg\} \end{aligned} \tag{3.11}$$

then we have

$$\rho_h(k) + \int_{-\infty}^{\infty} dk'\, g(k, k')\rho(k') = \rho^0(k) \tag{3.12}$$

The statistical interaction $g(k, k')$ contains two parts. The nontrivial one comes from the rates of change of the phase shifts with respect to momentum. Differing from the case of the *XXZ* chain with periodic boundary condition, two phase shifts rather than one contribute to the nontrivial part of the statistical interaction for a fixed momentum k. From equations (3.4) and (3.12), we can see that a dynamical equation can be translated into a statistical equation. This shows that the dynamical interaction can be transmuted into a statistical interaction.

REFERENCES

Alcaraz, F. C., *et al.* (1987). *Journal of Physics A: Mathematical and General*, **20**, 6397.
Bernard, D., and Wu, Y. S. (1994). Preprint SPhT-94-043, UU-HEP-94-03.
Haldane F. D. M. (1991). *Physical Review Letters*, **67**, 937.
Pasquier, V., and Saleur, H. (1990). *Nuclear Physics B*, **330**, 523.
Yang, C. N., and Yang, C. P. (1969). *Journal of Mathematical Physics*, **10**, 1115.